AN ANALYSIS OF THE TEACHER'S PROACTIVE ROLE IN SUPPORTING THE DEVELOPMENT OF SYMBOLIZATIONS

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The analysis reported in this paper documents the teacher's role in supporting the emergence of notational schemes from the students' problem-solving activity in one first-grade (age six) classroom. Initially symbolizations were offered by the teacher as a means of clarifying and communicating students' thinking. Later, the teacher worked to achieve her pedagogical agenda by using notational schemes to highlight certain solution processes. As a result, the introduction of notational schemes served to support shifts in the students' mathematical development.

Mathematical activity is communicated via a wide range of notations, schemas, and models that can be called *symbolizations*. These symbolizations are critical to the development of mathematical power in that they provide a vehicle for communication, representation, reflection, and argumentation. However, as Lesh, Post, and Behr (1987) observe, many students have "deficient understandings about the models and languages used to represent and manipulate mathematical ideas" (p. 37). This indicates the importance of supporting students' development of ways of symbolizing to communicate their mathematical reasoning. Further, if students are to develop grounded understandings of the meaning and use of these ways of modeling and symbolizing, it is essential that they view the symbols they use as descriptors of *their* mathematical activity. Students might then accept symbols and notations as ways of recording and communicating their thinking that they can use as the need arises. Ways of symbolizing should therefore emerge from students' informal mathematical activity and be consistent with their developing concepts and strategies (Cognition and Technology Group, 1990; Lesh & Akerstrom, 1982; Thompson, 1992). In addition, they should serve as a resource that students can use to describe, communicate, and reflect on their mathematical activity (Confrey, 1990; Kaput, 1987).

The purpose of this paper is to analyze the teacher's role in the emergence of ways of symbolizing and notating in one first-grade (age six) classroom. In doing so, I will document how the ways of symbolizing served as protocols of action as the students explained and justified their mathematical reasoning (Dörfler, 1989). In particular, the paper will focus on how the teacher's efforts in redescribing and notating students' explanations and solutions supported the emergence of ways of symbolizing from the students' activity. In addition to documenting the development of the ways of symbolizing and notating, the reported analysis will also relate the use of symbolizations both to shifts in discourse and to the students' development of their own ways of notating their reasoning. Although the particular emphasis is on the teacher's role, the analysis will necessarily document the interactive constitution of ways of symbolizing and notating as they emerged from classroom mathematical activity. In this way, the teacher's role can be characterized as that of supporting the emergence of both individual and collective ways of symbolizing.

The episodes reported in this paper are taken from a classroom in which I participated in a year-long teaching experiment in close collaboration with the teacher, Ms. Smith.[1] She, in fact, viewed the research team as peers with whom she reflected daily about numerous aspects of the mathematics class. One of the goals of the teaching experiment was to develop instructional sequences designed to address quantitative concepts typically introduced in first grade. In particular, the Patterning and Partitioning instructional sequence was designed to address early number concepts by providing students with opportunities to conceptually construct patterns and to partition collections of up to ten items (cf. McClain & Cobb, in press). The Structuring Numbers instructional sequence was intended to support students' ability to flexibly structure numbers in situations where they added and subtracted with sums and differences up to twenty (cf. Cobb, Gravemeijer, et al, 1997). A critical aspect of these sequences was developing ways to symbolize students' thinking that could then support shifts in their ability to reason in problem situations. As a result, Ms. Smith took a proactive role in initiating the use of symbols and notations to communicate students' ways of reasoning. In addition to providing a detailed account of the teacher's proactive role, the episodes will also clarify how the development of notational schemes became realized in the classroom by developing empirically grounded analyses of the teaching-learning process as it was interactively constituted in the classroom (Cobb, Wood, Yackel, & McNeal, 1992).

METHODOLOGY

Data were collected during the 1993-94 academic year and consist of daily videotape recordings of 103 mathematics lessons, copies of all the students' written work, daily field notes that summarize classroom events, and notes from daily debriefing sessions held with the project teacher. In addition, videotaped individual clinical interviews were conducted with each student in September, December, January, and May. A method described by Cobb and Whitenack (1996) for conducting longitudinal analyses of videotape sessions guided the analyses of the data. This method is consistent with Glaser and Strauss' (1967) constant comparative method for conducting ethnographic studies. It involves constantly comparing data as they are analyzed against conjectures and speculations generated thus far in the data analysis. As issues arose while viewing classroom videorecordings, they were documented and clarified through a process of conjecture and refutation.

The interpretive framework that guided the analysis is called the emergent perspective (cf. Cobb & Yackel, 1996). This framework emerged out of attempts to coordinate individual students' mathematical development with social processes as students' learning is accounted for in the social context of the classroom. It therefore places the students' and teacher's activity in social context by explicitly coordinating sociological and psychological perspectives. The psychological perspective is constructivist and treats mathematical development as a process of self-organization in which the learner reorganizes his or her activity in an attempt to achieve purposes or goals. The sociological perspective is interactionist and views communication as a process of mutual adaptation wherein individuals negotiate mathematical meanings. From this perspective, learning is characterized as the personal reconstruction of societal means and models through negotiation in interaction. Together, the two perspectives treat mathematical learning as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society.

CLASSROOM ANALYSIS

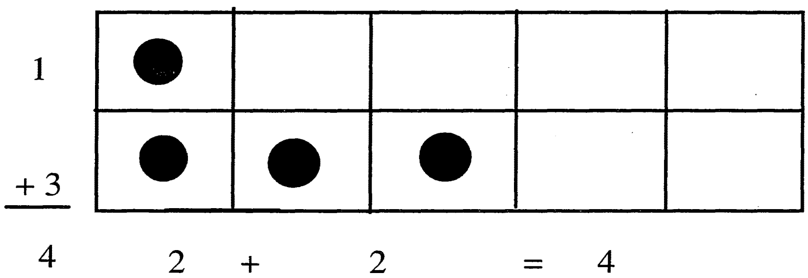
The ways of symbolizing that emerged in Ms. Smith's classroom appeared to evolve from the need to clarify and communicate students' thinking. At the beginning of the school year, students were not asked to make written records of their thinking but instead shared their solutions and explanations verbally in whole-class discussions. Ms. Smith judged that her students were unable to read well enough to be able to understand problems posed to them in text format. As a result of her desire to ensure that all students understood the task situation, she typically used the overhead projector or white board to pose tasks in a whole-class setting. This format was used throughout the Patterning and Partitioning instructional sequence and the first part of the Structuring Numbers instructional sequence. Students explained and justified their solutions verbally, often while relying on graphics from the overhead projector. During these activities, Ms. Smith would often introduce ways of symbolizing in an attempt to clarify a student's thinking either for herself or for other students. In contrast, during the second part of the Structuring Numbers sequence students completed individual activity sheets more frequently and were asked to make a record of their thinking so that others might understand their reasoning. These activities were subsequently discussed in the whole-class setting during which time Ms. Smith typically redescribed and notated students' explanations of their activity. As students participated in tasks from the instructional sequences, the use of notational schemes came to serve as thinking devices as students began to use the written records on the board as a means of reflecting on and comparing their own and other students' mathematical activity (Wertsch & Toma, 1995).

The Development of Number Sentences

In analyzing the development of number sentences in Ms. Smith's classroom, it is important to note that the use of *grouping* solutions emerged early in the school year as an accepted way to solve tasks (cf. McClain & Cobb, 1997). Grouping came to be interpreted as a solution process that involved grouping numbers or parts of numbers to quantify a collection as opposed to having to count the entire collection by ones. For instance, a student might identify the number of tiles in a domino pattern for six as *two groups of three* or as *a group of four and a group of two*. As Ms. Smith worked to support shifts in the students' ways of reasoning away from counting toward grouping, she would ask for *different* grouping ways to solve tasks. As a result, students learned to distinguish features of certain grouping solutions that caused them to be mathematically different. Although Ms. Smith still accepted counting solutions from those students whom she judged were currently unable to offer grouping solutions, grouping quantities to solve mathematical tasks became an accepted and valued way of solving tasks.

The emergence of number sentences as a means of recording the student's activity first occurred two weeks after the introduction of activities from the Patterning and Partitioning instructional sequence. Ms. Smith had developed the scenario of a pumpkin seller named Earl who sold his pumpkins in crates of ten. Using a single ten frame to represent Earl's crate, Ms. Smith posed problems by placing counters in the cells of the ten frame and asking students to determine how many pumpkins were in the crate (see Figure 1).

Figure 1
Ten Frame with Notation



Ms. Smith flashed (showed for three or four seconds on the overhead projector) a horizontal, single ten frame containing a certain number of chips and the students would tell how many chips/pumpkins they saw and how they saw them. Many responses built on the visual arrangement of the chips and were facilitated by the students' realization that each row in the crate held five pumpkins. However, it should be noted that some students attempted to count what they had seen by ones.

During one task, Ms. Smith flashed a horizontal, single ten frame containing four chips. In describing what she saw, Amy commented, "I saw three plus one." Ms. Smith then redescribed Amy's solution as "a group of three and a group of one" and notated it in vertical format to correspond with the placement of chips (see Figure 1). After the redescription and notation of Amy's explanation, Ms. Smith asked if anyone saw it a different way. Teri then responded that she saw "two going up and two going across." In response to Teri's explanation, Ms. Smith redescribed Teri's solution as "two groups of two" and notated as shown in Figure 1, careful to place the numerals so that, again, their placement corresponded with the placement of the chips. Through her notation, Ms. Smith implicitly indicated that she particularly valued both Amy's and Teri's solution method. The recasting of both explanations in terms of grouping further highlighted this fact. In addition, the number sentences provided a record for other students to use in reflecting on how Amy and Teri had each solved the task.

Throughout the Patterning and Partitioning instructional sequence, Ms. Smith regularly used conventional number sentences as a means of clarifying students' ways of reasoning. In this way, the notation served as a written description of the students' thinking. It is not clear what specific meanings the students may have given the number sentences. However, they appeared to be able to use these records as a way to compare and contrast their solution with ones that had already been offered. For this reason the number sentences supported students' reflection on their prior activity. This is indicated by the fact that in offering different ways to solve tasks, not only did students rarely duplicate previously offered solutions, but they also developed ways to clarify similarities and differences in theirs and others' solution processes. As a result, they were able to reason about features of the solution processes that allowed them to be interpreted as different. Further, their subsequent flexible use of number sentences indicates that they began to utilize the offered sentences as a way to think about and interpret their own strategies. The absence of individual student work and lack of direct evidence of the students' actual interpretation of number sentences prevent definitive claims. However, it is reasonable to suggest that students' participation in these classroom activities did give rise to learning opportunities which supported the development of number sentences as an accepted way of communicating mathematically.

As students continued to participate in activities from the Patterning and Partitioning instructional sequence during the next three weeks, an important shift occurred in the nature of the tasks in that the solution was no longer the total number of items (e.g. chips in the ten frame). The solution now entailed a process of grouping the collection in some way to find the total. In this way, students' prior activity of solving the task then became on object of reflection as they learned to judge for themselves what constituted a *different* solution. As a result of Ms. Smith's use of number sentences to communicate students' thinking, different was often interpreted in terms of how the solution might be symbolized. In other words, different solutions would require a different number sentence. This is a significant shift in that it now places the students' prior activity of solving the task in the background, diminishing the importance of simply finding the sum.

The Development of Notational Schemes

As students began participating in activities from the Structuring Numbers instructional sequence, the use of *going through ten* and *doubles* strategies emerged as taken-as-shared ways of solving tasks. For example, students often solved an addition task such as 7 + 8 by partitioning the eight into seven and one and reasoning, seven and seven is 14, and one more is 15. Ms. Smith devised a simple method of notating this reasoning by using an inverted "V" symbol that came to signify the partitioning or decomposing of a number. Ms. Smith would typically follow the "V" notation with the number sentences that expressed the result of the partitioning (see Figure 2).

Figure 2
Notating Decomposition of Numbers

$$
\begin{array}{c}
7 + 8 = \\
\quad\;\; / \;\; \backslash \\
\quad\;\; 7 \quad 1 \\
7 + 7 = 14 \\
14 + 1 = 15
\end{array}
$$

It could be argued that some of the students did not actually conceptually partition the eight, but instead reasoned 7 + 7 = 14, so one more is 15. This created a situation where students' counting-based solutions were expressed as collection-based solutions although neither is arguably more sophisticated. As a result, the "V" did not necessarily fit with the students' activity. However, Ms. Smith introduced the "V" notation and called it "splitting."

As an illustration, consider an incident that occurred on December 7. Students were asked to work individually on sheets composed of context problems supported by a graphic. Their task was to make a record of their solution processes so that others might understand their reasoning. The focus in this part of the lesson was on effectively communicating their mathematical thinking — not imitating a given notational system. During the subsequent whole-class discussion, Ms. Smith asked students to share their solution methods verbally as she redescribed and notated their activity. The first task to be discussed was: *there are eight people on the bus and six more get on.* The first offered solution involved using a doubles strategy.

> Kitty: I took one off the 8 and I put it on to the 6 to make 7 plus 7 and I know 7 plus 7 makes 14.

As she spoke, Ms. Smith redescribed and notated the solution as shown in Figure 3.

Figure 3
Teacher's Notation to Describe Kitty's Solution

8 + 6 =
/ \
7 1
7 + 7 = 14

After questions and discussion, Ms. Smith asked for a different way. Jane then explained that she partitioned the numbers differently.

> Jane: I stayed with the 6 but I broke it up into 3 and 3 and when it had the 3 it made 11 and 3 more... it made... uhm... it made 13 and one more is 14.

Again, Ms. Smith redescribed and notated the solution (see Figure 4), attempting to clarify to the students how Jane's explanation differed from Kitty's.

Figure 4
Notating Jane's Solution

8 + 6 =
/ \
3 3
8 + 3 = 11
11 + 3 = 14

Symbolizing the two solutions offered opportunities for the other students to clarify for themselves how these solutions compared to each other and to their own. In doing so, the students came to be able to judge for themselves if they had solved the task differently. This decision was, again, based in part on how they would envision their method being symbolized. As a result, the notational schemes continued to provide a means of highlighting the critical aspects of different solution strategies, even in new task situations.

In examining the written records the students made on the activity sheets for the task of *there are eight people on the bus and six more get on*, it is important to note that there was diversity in the students' notational schemes (see Figure 5).

Figure 5
Sample of Students' Notation Schemes

8 + 6	8 + 6	6 8	8 + 6 = 7
/ \	/ \	/ \	/ \
4 4	1 7	8 2 4 10 + 4 = 14	7 1
6 + 4 = 10			7 + 7 = 14
10 + 4 = 14	6 + 1 = 7 + 7 = 14	8 + 2 = 10	

Only the first student's way of notating is entirely consistent with that of the teacher. Although the other three students used elements of the teacher's scheme, they adapted them in original ways. Even when students' verbal explanations were redescribed and notated by Ms. Smith in a manner consistent with her original notational scheme, the students worked to devise notational schemes that expressed their thinking. As a consequence, although students might understand ways of talking about their activity that were expressed in terms of the teacher's notation scheme, they continued to solve tasks using a range of different, personally meaningful notation schemes.

As the instructional sequence progressed, Ms. Smith, on her own initiative, began to use the partitioning symbol "V" only if she judged that students had in fact partitioned an addend to arrive at the solution. It could be argued that in this way Ms. Smith was adapting to the students' solution methods; she stepped back from her own activity and reflected on the students' activity. An example can be seen in solutions to another task posed on December 7. The task is: *There are nine people on the bus and seven more get on.*

The first solution was offered by Anne.

> Anne: I thought about if I uhm. . . if I had ten and seven that would be seventeen and if I had nine and seven that would be sixteen. (Ms. Smith notates, see Figure 6).

Figure 6
Ms. Smith's Notating Anne's Solution

$$10 + 7 = 17$$
$$9 + 7 = 16$$

In this instance Ms. Smith judged that Anne had used the known relationship 10 + 7 = 17 and simply compensated by subtracting one. This did not imply a partitioning of the ten.

Next, Dan offered a partitioning solution which was notated with the "V" symbol as shown in Figure 7.

Figure 7
Ms. Smith's Notating Dan's Solution

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 9 + 7
    / \
   1   6
10 + 6 = 16
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> Dan: The way I did it was kinda like her (points to Anne's solution as notated by Ms. Smith). What I thought of is also ten. I have to take one out of the seven to make nine [sic] and I have the seven then is six so I had to make ten so. . . ten plus six is nineteen. . . sixteen.

In this episode, Ms. Smith judged Anne's solution to be different from Dan's. She discriminated between Dan's partitioning solution and Anne's use of a known relationship and subsequent counting. This is evidenced by the manner in which she notated these two solutions. Initially the students had to adapt to Ms. Smith's notational schemes. Here, Ms. Smith is adapting to the students.

It is important to note that Dan was able to use the written description of Anne's solution process to compare how his solution was not only like but also different from Anne's. This is evidenced by his opening statement as he clarified that he also tried to make ten. However, he judged his solution to be mathematically different. This was made possible by his reflection on and analysis of Anne's solution as supported by the notation. In this way, the notational schemes became thinking devices that supported reflective shifts in the students' activity.

CONCLUSION

Throughout this paper, we have attempted to document crucial aspects of Ms. Smith's proactive role in supporting her students' mathematical learning by guiding the development of ways of symbolizing and notating. For Ms. Smith, the notational schemes emerged from the students' attempts to explain and justify their thinking. They were not predetermined schemes imposed by the instructional sequence. The analysis of Ms. Smith's role in introducing these schemes indicates that while they became taken-as-shared, the teacher played a central role in initiating their development. In analyzing the classroom interactions, it appeared that Ms. Smith's use of ways of symbolizing provided a way for the students to organize and reflect on their activity—providing an opportunity for shifts in thinking to occur. These ways of symbolizing enabled reflection on and analysis of the students' prior mathematical activity. This is evidenced by the fact that on their own initiative, students often referred to the notation to explain their thinking to other children during whole-class discussions. In addition, students began to use the records as a means of comparing solutions, thereby initiating shifts in the discourse such that features of their reasoning became explicit topics of conversation. As such, the students' participation in such discourse supported their reflection on and mathematization of their prior activity.

However, it could be argued that these ways of symbolizing were imposed by the teacher to fit with her interpretation of the instructional tasks. Although the development of notational schemes used in Ms. Smith's classroom appeared to support shifts in the students' use of notation, the lack of explicit discussion about the students' notational schemes could have arguably served to delimit the extent of this shift. After students had worked individually on tasks, the subsequent whole-class discussions did not focus on the students' ways of representing their thinking. What typically transpired was that Ms. Smith copied a problem statement on the white board and asked the students to explain how they thought about it. Although numerous mathematically different interpretations were solicited from the students, each was represented by Ms. Smith using her notational scheme. In reflecting back on this aspect of classroom activities, the research team acknowledged the need for students to discuss their own notational schemes. Conversations with Ms. Smith during the classroom episodes had focused on her redescribing and notating student responses in an attempt to initiate shifts in mathematical discourse. In retrospect it appears that the students' role in developing notational schemes might have been brought to the fore more prominently. Further, it could be argued that in problematic situations that arose involving notation, asking students how they might notate the *problem* would have provided students an opportunity to play a larger role in the establishment of the notational schemes.

Nonetheless, Ms. Smith's proactive role in guiding the development of ways of notating appears to have been critical in supporting her students' mathematical development. The children increasingly notated on their own initiative as they solved problems while working both individually and in groups. These records helped them distance themselves from their ongoing activity and thus reflect on what they were doing. Consequently, the use of notation contributed to the productiveness of whole-class discussions by helping to make individual children's contributions explicit topics of conversation that could be compared and contrasted. It was as they participated in these discussions that the teacher guided her students' transition from informal, pragmatic problem solving to more sophisticated yet personally-meaningful mathematical activity.

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Endnote

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